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A CASE WHERE THE HAMILTON-JACOBI EQUATION IS INTEGRABLE

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Let us consider the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t}+\left(\frac{\partial S}{\partial r}\right)^2+F=0 \tag{1}$$

Here S is the action function and F is a given function of the variables r and t. We shall attempt to find the solution of this equation in the form

$$S=S_0+S_1, \quad S_0=X_0(x)T_0(t), \quad S_1=X_1(x)+T_1(t) \tag{2}$$

Here we have introduced the new variable $x=rf(t)$, where $f(t)$ is any doubly differentiable function. Substituting (2) into (1), we obtain

$$\frac{\partial S_0}{\partial t}+\frac{\partial S_0}{\partial x}\frac{x}{f(t)}\frac{df(t)}{dt}+\left(\frac{\partial S_0}{\partial x}\right)^2 f^2(t)+$$
$$+\frac{\partial S_1}{\partial x}\left[\frac{x}{f(t)}\frac{df(t)}{dt}+2\frac{\partial S_0}{\partial x}f^2(t)\right]+\frac{\partial S_1}{\partial t}+\left(\frac{\partial S_1}{\partial x}\right)^2 f^2(t)+F=0 \tag{3}$$

In the latter equation the coefficient of $\partial S_1/\partial x$ is equal to zero provided that

$$S_0=-\frac{x^2}{4f^3(t)}\frac{df(t)}{dt} \tag{4}$$

For this S_0 Eq. (3) yields

$$x^2\left[\frac{1}{2f(t)^6}\left(\frac{df(t)}{dt}\right)^2-\frac{1}{4f(t)^5}\frac{d^2f(t)}{dt^2}\right]+\frac{dT_1(t)}{dt}\frac{1}{f(t)^2}+\left(\frac{dX_1(x)}{dx}\right)^2+\frac{1}{f(t)^2}F=0 \tag{5}$$

The variables in this equation are separable if

$$F=f^2(t)\Psi(x)+f^2(t)\eta(t)+x^2\left[\frac{1}{4f^3(t)}\frac{d^2f(t)}{dt^2}-\frac{1}{2f(t)^4}\left(\frac{df(t)}{dt}\right)^2\right] \tag{6}$$

Here $\Psi(x)$ and $\eta(t)$ are arbitrary functions. If this condition is fulfilled, the total integral Eq. (1) is

$$S=-\frac{x^2}{4f^3(t)}\frac{df(t)}{dt}\pm\int\sqrt{C_1-\Psi(x)}\,dx-\int f^2(t)(C_1+\eta(t))\,dt+C_2$$

where C_1 and C_2 are arbitrary constants.

Specifically, separation of variables in Eq. (5) is possible if

$$\frac{1}{2f^6(t)}\left(\frac{df}{dt}\right)^2-\frac{1}{4f^5(t)}\frac{d^2f}{dt^2}=k, \qquad f=\frac{1}{\sqrt{a(t-b)^2+4k/a}} \qquad F=f^2(t)X(x) \tag{7}$$

Here $X(x)$ is an arbitrary function; k, a and b are constants.

The solution of Eq. (1) under conditions (7) is obtained in [1] by the Liapunov-Charpy method [2].

By virtue of the Jacobi theorem [3], the resulting total integral (1) can be used to find the solution of the associated canonical system of differential equations. This system yields the differential Eq.
$$\frac{d^2r}{dt^2} = -2\frac{\partial F}{\partial r}$$

Thus, the latter equation can be solved for a function F satisfying condition (6).

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CYCLES AND QUASI-INDICES OF SINGULAR POINTS OF CONSERVATIVE SYSTEMS

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Henri Poincaré [1] noted that closed trajectories (cycles) investigated, in the whole, play a role roughly analogous to that of singular points in the study of the behavior of trajectories in the small.

However, the problem of finding the cycles is in itself quite difficult. Among the criteria of existence of periodic trajectories for two-dimensional systems we must first of all note the criteria based on a consideration of vector field rotation (the indices of the Poincaré singular points).

A sufficient criterion for the existence of periodic trajectories on a plane based on the so-called ring principle whereby the velocity vector on the boundary of the domain is everywhere directed into or out of the ring was pointed out by Bendixon [2 and 3].

There exist still other methods of investigation in the whole, among them the method of Liapunov functions [4].

The criterion of existence of periodic trajectories for conservative systems in the so-called invertible case, which is based on a consideration of the variation of the action integral, was set forth by Whittaker [5]. Our study of cycles for conservative systems is based on a different principle, and specifically on the study of quasi-indices as structural